ISRAEL JOURNAL OF MATHEMATICS, Vol. 52, No. 4, 1985

SOME RESULTS ON CONSECUTIVE LARGE CARDINALS II: APPLICATIONS OF RADIN FORCING†

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ABSTRACT

Let κ be a 3 huge cardinal in a countable model V of ZFC, and let A and B be subsets of the successor ordinals $< \kappa$ so that $A \cup B = \{\alpha < \kappa : \alpha$ is a successor ordinal$\}$. Using techniques of Gitik, we construct a choiceless model N_A of ZF of height κ so that $N_A \models$ "ZF + $\neg AC_\omega$ + For $\alpha \in A$, $\aleph_\alpha$ is a Ramsey cardinal + For $\beta \in B$, $\aleph_\beta$ is a singular Rowbottom cardinal which carries a Rowbottom filter + For γ a limit ordinal, $\aleph_\gamma$ is a Jonsson cardinal which carries a Jonsson filter".

Radin forcing is one of the most powerful tools currently being used by set theorists. Its applications are both well known and extensive, as witnessed by the work of Woodin [15], Woodin and Foreman [3], Mitchell [11], Gitik [4], and others.

This paper presents a further application of Radin forcing to the construction of choiceless models of ZF. Specifically, the following theorem is proven.

THEOREM 1. *Let* $V \models$ "ZFC + κ *is a* 3 *huge cardinal* + A *and* B *are disjoint subsets of the successor ordinals* $< \kappa$ *so that* $A \cup B = \{\alpha < \kappa : \alpha$ *is a successor ordinal*$\}$". *There is then a model* N_A *of* ZF + $\neg$AC (*in fact, of* ZF + $\neg AC_\omega$) *whose ordinals have height* κ *so that* $N_A \models$ "*For* $\alpha \in A$, $\aleph_\alpha$ *is a Ramsey cardinal* + *For* $\beta \in B$, $\aleph_\beta$ *is a singular Rowbottom cardinal which carries a Rowbottom filter* + *For* γ *a limit ordinal,* $\aleph_\gamma$ *is a Jonsson cardinal which carries a Jonsson filter*".

Note that Theorem 1 has one of its consequences that if $A = \{\alpha < \kappa : \alpha$ is a successor ordinal$\}$ and $B = \emptyset$, then $N_A \models$ "ZF + Every successor cardinal is a

† The author wishes to express his thanks to the Rutgers Research Council for a Summer Research Fellowship which partially supported this work.

The author also wishes to thank Moti Gitik and Bob Mignone for their useful comments concerning the subject matter of this paper.

Received May 10, 1984 and in revised form February 25, 1985

Ramsey cardinal + Every limit cardinal is a Jonsson cardinal". This extends Theorem 1 of [1], although for the model N of [1], $N \models \mathrm{DC}$.

The proof of Theorem 1 relies heavily on Gitik's techniques of [4] and on the techniques of [1]. (Gitik has pointed out that an almost huge cardinal suffices to prove Theorem 1. See the end of this paper for further details.) Before beginning the proof of Theorem 1, however, some preliminary material will be discussed.

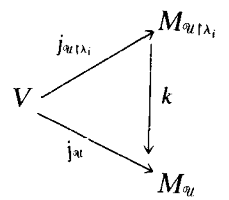
The notation and terminology used in this paper are fairly standard. For x a set, $\mathrm{TC}(x)$ denotes the transitive closure of x, $|x|$ denotes the cardinality of x, and $\bar{x}$ denotes the order type of x. If α and β are ordinals, then $[\alpha,\beta]$, $[\alpha,\beta)$, $(\alpha,\beta]$, and (α,β) are as in standard interval notation. $R(\kappa)$ denotes the sets of rank $<\kappa$. For κ and λ cardinals $\kappa\le\lambda$, $P_\kappa(\lambda)=\{x\subseteq\lambda: |x|<\kappa\}$ and $P^\kappa(\lambda)=\{x\subseteq\lambda: |x|=\kappa\}$.

We assume complete familiarity with supercompact cardinals. For $n<\omega$, a cardinal κ is said to be n huge iff there is a cardinal $\lambda>\kappa$, a κ additive normal ultrafilter $\mathcal{U}$ on 2^λ, and a sequence $\kappa=\lambda_0<\lambda_1<\cdots<\lambda_n=\lambda$ so that for each $i<n$, $\{x\subseteq\lambda: \overline{x\cap\lambda_{i+1}}=\lambda_i\}\in\mathcal{U}$. It is easily seen that $\mathcal{U}$ concentrates on $P^{\lambda_{n-1}}(\lambda_n)$. For $C\in\mathcal{U}$ and $p\in C$ let $C\restriction\lambda_i=\{p\cap\lambda_i: p\in C\}$. It is also easily seen that for each $0\le i<n$, $\mathcal{U}\restriction\lambda_i=\{C\restriction\lambda_i: C\in\mathcal{U}\}$ is a κ additive normal ultrafilter on $P^{\lambda_{i-1}}(\lambda_i)$ and that $\mathcal{U}\restriction\lambda_i$ witnesses the i hugeness of κ with the sequence $\langle\lambda_j: j\le i\rangle$. $\mathcal{U}$ will be referred to as an n huge ultrafilter on $P^{\lambda_{n-1}}(\lambda_n)$ with sequence $\langle\lambda_i: i\le n\rangle$, and for $0\le i<n$, $\mathcal{U}\restriction\lambda_i$ will be referred to as the restriction ultrafilter to λ_i. A 1 huge cardinal will be called simply a huge cardinal.

It is well known (see [5]) that the above definition of n hugeness is equivalent to the existence of an elementary embedding $j: V\to M$, M a transitive inner model of ZFC so that $M^{j^n(\kappa)}\subseteq M$, where $j^n(\kappa)$ is the ordinal obtained by n successive applications of j to κ, i.e., for $0\le i\le n$, $j^0(\kappa)=\kappa$ and $j^i(\kappa)=j(j^{i-1}(\kappa))$. Further, if $\mathcal{U}$ is an n huge ultrafilter $\mathcal{U}$ on $P^{\lambda_{n-1}}(\lambda_n)$ with sequence $\langle\lambda_i: i\le n\rangle$, the embedding j and the inner model M are obtained in the usual manner by forming the ultrapower $V^{P^{\lambda_{n-1}}(\lambda_n)}/\mathcal{U}$ and taking its transitive collapse. The sequence $\langle\lambda_i: i\le n\rangle$ will be so that $\lambda_i=j^i(\kappa)$. We will always assume that for $j: V\to M$ an embedding which witnesses the n hugeness of κ, both j and M are generated by an n huge ultrafilter $\mathcal{U}$ on $P^{\lambda_{n-1}}(\lambda_n)$ with sequence $\langle\lambda_i: i\le n\rangle$.

The embedding j associated with the ultrafilter $\mathcal{U}$ will sometimes be written as $j_{\mathcal{U}}$, and the associated inner model will sometimes be written as $M_{\mathcal{U}}$. Note further that for $\mathcal{U}$ a huge ultrafilter on $P^\kappa(\lambda_1)$, $\alpha\in[\kappa,\lambda)$, if we let, for $C\in\mathcal{U}$, $C\restriction\alpha=\{p\cap\alpha: p\in C \text{ and } p\cap\alpha\in P_\kappa(\alpha)\}$ then $\mathcal{U}\restriction\alpha=\{C\restriction\alpha: C\in\mathcal{U}\}$ is a normal measure on $P_\kappa(\alpha)$.

We will frequently confuse an ultrapower with its transitive collapse. Keeping this in mind, let us note that if $\mathcal{U}$ is an n huge ultrafilter on $P^{\lambda_{n-1}}(\lambda_n)$ with sequence $\langle \lambda_i : i \leq n \rangle$, then any ordinal $\alpha \leq \lambda_n$ is represented in $M_{\mathcal{U}}$ by $[\overline{p \cap \alpha}]_{\mathcal{U}}$; this parallels the fact that for any normal ultrafilter $\mathcal{U}$ on $P_\kappa(\lambda)$, any ordinal $\alpha \leq \lambda$ is represented in $M_{\mathcal{U}}$ by $[\overline{p \cap \alpha}]_{\mathcal{U}}$. Note also that, in analogy to the case of supercompact cardinals (see [10]), for any $\mathcal{U} \restriction \lambda_i$ we have the commutative diagram



where for $[f]_{\mathcal{U}\restriction\lambda_i} \in M_{\mathcal{U}\restriction\lambda_i}$, g a representative of $[f]_{\mathcal{U}\restriction\lambda_i}$, $h: P^{\lambda_{n-1}}(\lambda_n) \to V$ defined by $h(p) = g(p \cap \lambda_i)$, $k([f]_{\mathcal{U}\restriction\lambda_i}) = [h]_{\mathcal{U}}$. The fact that k is a well defined elementary embedding from $M_{\mathcal{U}\restriction\lambda_i}$ into $M_{\mathcal{U}}$ is easily verified. It is also easily shown that for any ordinal $\alpha \leq \lambda_i$, $k(\alpha) = \alpha$. Since each λ_i is strongly inaccessible in V, the preceding fact has an immediate consequence that for any $x \in V$ so that $|\mathrm{TC}(x)| < \lambda_i$, $x \in M_{\mathcal{U}\restriction\lambda_i}$ and $k(x) = x$.

Our forcing terminology is also fairly standard. We always assume that the ground model V is countable so that generic objects can be produced. For P a partial ordering, $\Vdash_P \phi$ means that the empty condition (weakly) forces ϕ. Terms in the forcing language associated with P are indicated by $\tilde{x}$, $\check{x}$, or $\underset{\sim}{x}$. We beg the question of the meaning of $\leq$ by avoiding its usage and saying that q ext p means q contains more information than p.

Two partial orderings will be crucial in the proof of Theorem 1, namely the Lévy collapse and supercompact Prikry forcing. If $\kappa < \lambda$ are regular cardinals, the Lévy collapse ordering

$$\mathrm{Col}(\kappa,\lambda) = \{p : p : \kappa \times \lambda \to \lambda \text{ is a function so that } |\mathrm{dmn}(p)| < \kappa \text{ and so that } p(\langle \alpha, \beta \rangle) < \beta\},$$

with q ext p if $q \supseteq p$. For $\alpha \in (\kappa, \lambda)$ a regular cardinal, $p \in \mathrm{Col}(\kappa,\lambda)$,

$$p \restriction \alpha = \{\langle\langle \beta, \gamma \rangle, \delta\rangle \in p : \gamma < \alpha\} \quad \text{and} \quad \mathrm{Col}(\kappa,\lambda)\restriction\alpha = \{p\restriction\alpha : p \in \mathrm{Col}(\kappa,\lambda)\}.$$

If G is V-generic on $\mathrm{Col}(\kappa,\lambda)$, then it is easily seen that $G\restriction\alpha = \{p\restriction\alpha : p \in G\}$ is V-generic on $\mathrm{Col}(\kappa,\lambda)\restriction\alpha = \mathrm{Col}(\kappa,\alpha)$.

Assume now that $\kappa < \lambda$ are cardinals and κ is λ supercompact. Let $\mathcal{U}$ be a normal measure on $P_\kappa(\lambda)$ which has the Menas partition property [9]. Supercompact Prikry forcing on $P_\kappa(\lambda)$,

$$\mathrm{SC}(\kappa,\lambda)=\{\langle p_1,\ldots,p_n,C\rangle:\ p_i\in P_\kappa(\lambda),\ \ C\in\mathcal{U},\ \ p_i\subsetneq p_j\ \text{ for } 1\le i<j\le n,\ \text{ and } q\in C\Rightarrow \textstyle\bigcup_{i=1}^n p_i\subsetneq q\},$$

where $p\subsetneq q$ means $p\subseteq q$ and $\bar{p}<\overline{q\cap\kappa}$. For $\pi,\pi'\in\mathrm{SC}(\kappa,\lambda)$, $\pi=\langle p_1,\ldots,p_n,C\rangle$, $\pi'=\langle q_1,\ldots,q_m,D\rangle$, π' ext π if the following four conditions hold:

(1) $n\le m$.

(2) For $i\le n$, $p_i=q_i$.

(3) For $n<i\le m$, $q_i\in C$.

(4) $D\subseteq C$.

Note that if $\lambda\ge 2^\kappa$, then without loss of generality we can assume that for the condition $\langle p_1,\ldots,p_n,C\rangle$, each p_i and each $q\in C$ is so that $\overline{p_i\cap\kappa}$ and $\overline{q\cap\kappa}$ are measurable cardinals.

For $\alpha\in[\kappa,\lambda)$ a regular cardinal,

$$\pi=\langle p_1,\ldots,p_n,C\rangle\in\mathrm{SC}(\kappa,\lambda),\qquad \pi\upharpoonright\alpha=\langle p_1\cap\alpha,\ldots,p_n\cap\alpha,C\upharpoonright\alpha\rangle,$$

where $C\upharpoonright\alpha$ is defined as in the earlier discussion on n huge cardinals. Let $\mathrm{SC}(\kappa,\lambda)\upharpoonright\alpha=\{\pi\upharpoonright\alpha:\ \pi\in\mathrm{SC}(\kappa,\lambda)\}$. If G is V-generic on $\mathrm{SC}(\kappa,\lambda)$, then it is again easily seen that $G\upharpoonright\alpha=\{\pi\upharpoonright\alpha:\ \pi\in G\}$ is V-generic on $\mathrm{SC}(\kappa,\lambda)\upharpoonright\alpha=\mathrm{SC}(\kappa,\alpha)$, where $\mathrm{SC}(\kappa,\alpha)$ is defined using $\mathcal{U}\upharpoonright\alpha=\{C\upharpoonright\alpha:\ C\in\mathcal{U}\}$, a normal measure on $P_\kappa(\alpha)$ with the Menas partition property.

We briefly review the definition of Ramsey, Rowbottom, and Jonsson cardinals. We assume complete familiarity with the Erdös partition notation. A cardinal κ is Ramsey if $\kappa\to(\kappa)^{<\omega}$. κ is Rowbottom if $\forall\lambda<\kappa[\kappa\to[\kappa]^{<\omega}_{\lambda,\omega}]$. κ is Jonsson if $\kappa\to[\kappa]^{<\omega}_\kappa$. A cardinal κ carries a Ramsey, Rowbottom, or Jonsson filter $\mathcal{U}$ if every Ramsey, Rowbottom, or Jonsson partition has a homogeneous set in $\mathcal{U}$. For further information on these cardinals, consult [5] or [6].

Finally, if a notion ϕ is not absolute, then ϕ^V will mean the notion ϕ in the model V.

We turn now to the proof of Theorem 1. Where convenient, we will adopt Gitik's notation of [4]. First, however, we prove a preliminary result which will be used in the construction of the forcing conditions.

THEOREM 1.1. *Let $V\models$ "ZFC + κ is a 3 huge cardinal", and let j: $V\to M$ be an elementary embedding which witnesses the 3 hugeness of κ, with λ_0, λ_1, λ_2, and λ_3 as in the preliminaries. There is then a generic extension $V[G]$ of C with the following properties*:

(1) *κ is huge with a normal measure $\mathcal{U}_0$ on $P^\kappa(\lambda_1)$.*

(2) *The* $<\lambda_1$ *supercompactness of* κ *is indestructible under forcing with* κ *directed closed partial orderings* Q *so that* $|\mathrm{TC}(Q)|<\lambda_1$.

The proof of Theorem 1.1 uses the following lemma which is stated in terms of n hugeness for $n\geq 3$.

LEMMA 1.2. *Let* $j\colon V\to M$ *be an elementary embedding which witnesses the* n *hugeness of* κ, *with* $\langle\lambda_i\colon 0\leq i\leq n\rangle$ *as in the preliminaries. There is then an* $f\colon \kappa\to R(\kappa)$ *such that for every* x *so that* $|\mathrm{TC}(x)|<\lambda_1$, *there is an* $n-1$ *huge ultrafilter* $\mathcal{U}_x$ *on* $P^{\lambda_{n-2}}(\lambda_{n-1})$ *with sequence* $\langle\lambda_i\colon 0\leq i\leq n-1\rangle$ *so that* $(j_{\mathcal{U}_x}f)(\kappa)=x$.

PROOF OF LEMMA 1.2. The proof of this lemma involves a slight modification of Laver's argument of [7]. First, suppose that the conclusion of the lemma is false. There is then for each $f\colon \kappa\to R(\kappa)$ an x so that $|\mathrm{TC}(x)|<\lambda_1$ and so that for all $n-1$ huge measures $\mathcal{U}$ on $P^{\lambda_{n-2}}(\lambda_{n-1})$ with sequence $\langle\lambda_i\colon 0\leq i\leq n-1\rangle$, $(j_{\mathcal{U}}f)(\kappa)\neq x$. Thus, if we let $\phi(f,\alpha_0,\alpha_1,\ldots,\alpha_{n-1})$ be the statement "α_0 is a cardinal and $f\colon \alpha_0\to R(\alpha_0)$ is a function so that for some x with $|\mathrm{TC}(x)|<\alpha_1$ and all α_0 additive $n-1$ huge measures $\mathcal{U}$ on $P^{\alpha_{n-2}}(\alpha_{n-1})$ with sequence $\langle\alpha_i\colon 0\leq i\leq\alpha_{n-1}\rangle$, $(j_{\mathcal{U}}f)(\alpha_0)\neq x$", then by the closure properties of M, $M\models\phi(f,\kappa,\lambda_1,\ldots,\lambda_{n-1})$ for each $f\colon \kappa\to R(\kappa)$. Hence, for μ the normal measure on κ generated by j, $A_0=\{\alpha<\kappa\colon \alpha$ is an $n-1$ huge cardinal so that for all α additive $n-1$ huge measures $\mathcal{U}$ on $P^{\alpha_{n-2}}(\alpha_{n-1})$ with sequence $\langle\alpha,\langle\alpha_i\colon 1\leq i\leq n-1\rangle\rangle$ and all $f\colon \alpha\to R(\alpha)$ there is an x so that $|\mathrm{TC}(x)|<\alpha$ and $(j_{\mathcal{U}}f)(\alpha)\neq x\}\in\mu$, where $\langle\alpha_1,\alpha_2,\ldots,\alpha_{n-1}\rangle=\langle\kappa,\lambda_1,\ldots,\lambda_{n-2}\rangle$.

As Laver does in [7], we inductively define a function $f\colon \kappa\to R(\kappa)$. Let $f_\alpha=f\restriction\alpha$, and if $\alpha\notin A_0$, let $f(\alpha)=\varnothing$. If $\alpha\in A_0$ and $f\colon \alpha\to R(\alpha)$, then let x_α be a set which witnesses $\phi(f_\alpha,\alpha,\alpha_1,\ldots,\alpha_{n-1})$ and set $f(\alpha)=x_\alpha$; otherwise, again let $f(\alpha)=\varnothing$. It is then the case that $j(\langle f_\alpha\colon \alpha\in A_0\rangle)\restriction\kappa=(jf)_\kappa=f$ and $j(\langle x_\alpha\colon \alpha\in A_0\rangle)(\kappa)=$ an x which witnesses $\phi(f,\kappa,\lambda_1,\ldots,\lambda_{n-1})$ in M and hence also in V.

Let $\mathcal{U}'$ be the n huge ultrafilter on $P^{\lambda_{n-1}}(\lambda_n)$ with sequence $\langle\lambda_i\colon i\leq n\rangle$ which generates j, and let $k\colon M_{\mathcal{U}'\restriction\lambda_{n-i}}\to M_{\mathcal{U}'}=M$ be the canonical elementary embedding. From the canonical diagram and the facts that k is the identity on λ_{n-1}, $M^{\lambda_1}_{\mathcal{U}'\restriction\lambda_{n-1}}\subseteq M_{\mathcal{U}'\restriction\lambda_{n-1}}\subseteq M_{\mathcal{U}'}$, and $|\mathrm{TC}(x)|<\lambda_1$ we have that $x\in M_{\mathcal{U}'\restriction\lambda_{n-1}}$ and $k(x)=x$. Thus,

$$(j_{\mathcal{U}'\restriction\lambda_{n-1}}f)(\kappa)=k^{-1}((j_{\mathcal{U}'}f)(\kappa))=k^{-1}(x)=x.$$

This contradicts the hypothesis that for every $n-1$ huge measure $\mathcal{U}$ on $P^{\lambda_{n-2}}(\lambda_{n-1})$ with sequence $\langle\lambda_i\colon 0\leq i\leq n-1\rangle$, $(j_{\mathcal{U}}f)(\kappa)\neq x$, thus proving Lemma 1.2. □ Lemma 1.2

We return now to the proof of Theorem 1.1. Since we are assuming that κ is 3 huge, by Lemma 1.2 let $f\colon \kappa\to R(\kappa)$ be a function so that for any x with $|\mathrm{TC}(x)|<\lambda_1$ there exists a 2 huge measure $\mathcal{U}_x$ on $P^{\lambda_1}(\lambda_2)$ with sequence $\langle\lambda_0,\lambda_1,\lambda_2\rangle$ so that $(j_{\mathcal{U}_x}f)(\kappa)=x$. The ultrafilter $\mathcal{U}_x$ which will be used will be for the set x which is a term in the appropriate partial ordering for $\langle \mathrm{Col}((2^\kappa)^+,\eta),\kappa\rangle$, where η is the least strongly inaccessible cardinal $>\kappa$. The set x will serve as a "coding set" for a portion of the Laver partial ordering by which we force in order to obtain the model for Theorem 1.1. It will be the case that $\mathrm{Col}((2^\kappa)^+,\eta)$ can be replaced by any suitable non-trivial partial ordering which is at least $(2^\kappa)^+$ directed closed.

Define in κ stages a Laver partial ordering P^0 using f, at each stage α choosing an ordinal γ_α, as follows. $P_0=\{\varnothing\}$, and $\gamma_0=0$. If λ is a limit ordinal, then P_λ consists of those elements of inverse limit $(\langle P_\alpha\colon \alpha<\lambda\rangle)$ whose supports are the appropriate Easton set of ordinals, and $\gamma_\lambda=\bigcup_{\alpha<\lambda}\gamma_\alpha$. To define $P_{\alpha+1}$, let $\sigma_\alpha=\bigcup\{\delta\colon \delta$ is an ordinal so that for some $\gamma<\alpha$, $P_{\gamma+1}=P_\gamma*\tilde{Q}_\gamma$ and $\Vdash_{P_\gamma}$ "$Q_\gamma\neq\{\varnothing\}$ and Q_γ is δ directed closed"$\}$. $P_{\alpha+1}=P_\alpha*\tilde{Q}_\alpha$, where $\tilde{Q}_\alpha$ is a term for $\{\varnothing\}$ and $\gamma_{\alpha+1}=\gamma_\alpha$ unless for all $\beta<\alpha$, $\gamma_\beta<\alpha$ and $f(\alpha)=\langle\tilde{Q},\zeta\rangle$ for ζ an ordinal and $\tilde{Q}$ a term so that $\Vdash_{P_\alpha}$ "Q is a partial ordering which is at least $\max(\alpha,\sigma_\alpha)$ directed closed"; in this case, $\tilde{Q}_\alpha=\tilde{Q}$ and $\gamma_\alpha=\zeta$. Finally, let $P^0=P_\kappa=$ those elements of the inverse limit of $\langle P_\alpha\colon \alpha<\kappa\rangle$ whose supports are the appropriate Easton set of ordinals.

Let now $x=\langle\widetilde{\mathrm{Col}}((2^\kappa)^+,\eta),\kappa\rangle$, where $\widetilde{\mathrm{Col}}((2^\kappa)^+,\eta)$ is a term for $\mathrm{Col}((2^\kappa)^+,\eta)$ in the forcing language associated with P^0, and let $\mathcal{U}_x$ be the 2 huge ultrafilter on $P^{\lambda_1}(\lambda_2)$ with sequence $\langle\lambda_0,\lambda_1,\lambda_2\rangle$ discussed before. Call $j_{\mathcal{U}_x}k$ and $M_{\mathcal{U}_x}N$. Note again that $k(\kappa)=\lambda_1$ and $k(\lambda_1)=\lambda_2$; hence, in N, $k(P^0)$ is a partial ordering defined in $k(\kappa)=\lambda_1$ stages using the function $k(f)=g$ in the manner specified in the previous paragraph. Since k is the identity on κ and $f\colon \kappa\to R(\kappa)$, $g\restriction\kappa=f$, so by the closure properties of N, $k(P^0)\restriction\kappa=P^0$. Further, since we can assume that $\kappa=\bigcup_{\alpha<\kappa}\gamma_\alpha$, the choice of f insures that in N, $P_{\kappa+1}=P_\kappa*\widetilde{\mathrm{Col}}((2^\kappa)^+,\eta)$, and the definition of $k(P^0)$ in N insures that in N, for P^1 the partial ordering so that $k(P^0)=P_\kappa*\tilde{P}^1$, $\Vdash_{P_\kappa}$ "P^1 is $(2^\kappa)^+$ directed closed". By the closure properties of N, in V $\Vdash_{P^0}$ "P^1 is $(2^\kappa)^+$ directed closed". Let $P=k(P^0)$. P can be defined in V in λ_1 stages in the manner specified in the preceding paragraph using the function g, and P can be written in V as P^0*P^1.

Let G^0 be V-generic on P^0. We show the existence of a $V[G^0]$-generic set G^1 on P^1 so that $k''G^0\subseteq G^0*G^1$. As $|P^0|\le 2^\kappa$, the closure properties of N imply that $k''P^0\in N$, so $k''G^0\in N[G^0]$. For each $p\in P^0$, $k(p)\in P^0*\tilde{P}^1=P$ is a condition defined in λ_1 steps in a manner analogous to p. Using again the fact that k is the identity on $R(\kappa)$, we can write $k(p)$ as $p*p^1$ where $p^1 \in P^1$. Since $|k''G^0| \leq 2^\kappa$ in either $V[G^0]$ or $N[G^0]$, we can use the facts that $\Vdash_{P^0}$ "P^1 is $(2^\kappa)^+$ directed closed" and $k''G^0$ is a compatible set of conditions to produce (in either $V[G^0]$ or $N[G^0]$) an upper bound q^0 to the compatible set of conditions $\{p^1: p \in G^0\}$. Let G^1 be a $V[G^0]$-generic subset of P^1 which contains q^0. If $p \in G^0$, then $k(p) = p*p^1$, so since q^0 ext p^1, $p^1 \in G^1$. Thus, $k''G^0 \subseteq G^0 * G^1$.

Let $G = G^0 * G^1$. We now show that $V[G] \models$ "κ is huge with a normal measure $\mathcal{U}_0$ on $P^\kappa(\lambda_1)$". The proof of this fact will be by a slight modification of Silver's original argument. First, we show that in N, P is an initial segment of $k(P)$. In V, $P = P^0 * \tilde{P}^1$, so in N, $k(P) = k(P^0) * k(\tilde{P}^1)$. By our earlier work, $N \models$ "$k(P^0) = P^0 * \tilde{P}^1$", and $V \models$ "$P = P^0 * \tilde{P}^1$". Thus, since $k(P^0)$ is in N the portion of $k(P)$ defined through stage λ_1, and $k(\tilde{P}^1)$ is in N a term for the portion of $k(P)$ defined between stages λ_1 and λ_2, in N P is an initial segment of $k(P)$.

We next show the existence of a $V[G]$-generic set H on $k(P^1)$ so that $k''G \subseteq G * H$. As $|P| \leq 2^{\lambda_1}$, the closure properties of N again imply that $k''G \in N[G]$. If we write each $p \in P$ as $p^0 * p^1$, where $p^0 \in P^0$ and $p^1 \in P^1$, then $k(p) = k(p^0) * k(p^1)$, where $k(p^0) \in P^0 * \tilde{P}^1 = P$ and $k(p^1) \in k(P^1)$. By the definition of P and the closure properties of N, in N and $V \Vdash_P$ "$k(P^1)$ is $(2^{\lambda_1})^+$ directed closed". Thus, as before, $\{k(p^1): p \in G\}$ has an upper bound q in either $V[G]$ or $N[G]$. Let H be a $V[G]$-generic subset of $k(P^1)$ which contains q. If $p = p^0 * p^1 \in G = G^0 * G^1$, then by our earlier construction, $k(p^0) \in G^0 * G^1$, and as q ext p^1, $p^1 \in H$. Thus, $k''G \subseteq G * H$.

Using the generic sets G and $G * H$, we define an elementary embedding $\bar{k}$: $V[G] \to N[G*H]$ which extends k by $\bar{k}(i_G(\tau)) = i_{G*H}(k(\tau))$, where τ is a term in the forcing language associated with P. The proof that $\bar{k}$ is well defined is as in Silver's original argument, and the proof that $\bar{k}$ is elementary is by induction on the length of formulae and uses the fact that $k''G \subseteq G * H$. We can now use k and $\bar{k}$ to define a huge ultrafilter $\mathcal{U}_0$ on $(P^\kappa(\lambda_1))^{V[G]}$ by $C \in \mathcal{U}_0 \Leftrightarrow C \in P^\kappa(\lambda_1)$ and $\langle k(\alpha): \alpha < \lambda_1 \rangle \in \bar{k}(C)$. The proof that $\mathcal{U}_0$ is a huge ultrafilter on $P^\kappa(\lambda_1)$ is standard (see [5], p. 198). By the fact that $k(P^1)$ is $(2^{\lambda_1})^+$ directed closed, $\mathcal{U}_0 \in V[G]$.

Finally, we show that $V[G] \models$ "The $< \lambda_1$ supercompactness of κ is indestructible under forcing with κ directed closed partial orderings Q so that $|\mathrm{TC}(Q)| < \lambda_1$". Let Q be such a partial ordering in $V[G]$, and let $\gamma < \lambda_1$ be $|\mathrm{TC}(Q)|$. By elementariness, $N \models$ "g: $\lambda_1 \to R(\lambda_1)$ is a function so that for any x with $|\mathrm{TC}(x)| < \lambda_2$ there is a 2 huge measure $\mathcal{U}_x$ on $P^{\lambda_2}(k(\lambda_2))$ with sequence $\langle \lambda_1, \lambda_2, k(\lambda_2) \rangle$ so that $(j_{\mathcal{U}_x} g)(\lambda_1) = x$"; thus, it follows that $N \models$ "λ_1 is the least ordinal moved by $j_{\mathcal{U}_x}$". In N, let x be $\langle \widetilde{\mathrm{Col}}(k(\gamma)^+, \delta), \lambda_1 \rangle$, where δ is the least strongly inaccessible cardinal $> k(\gamma)$ and $\widetilde{\mathrm{Col}}(k(\gamma)^+, \delta)$ is a term in the forcing language associated with P for $\mathrm{Col}(k(\gamma)^+, \delta)$ and let $\mathcal{U}_x$ be a 2 huge ultrafilter on $P^{\lambda_2}(k(\lambda_2))$ with sequence $\langle \lambda_1, \lambda_2, k(\lambda_2) \rangle$ so that $(j_{\mathcal{U}_x} g)(\lambda_1) = \langle \widetilde{\mathrm{Col}}(k(\gamma)^+, \delta), \lambda_1 \rangle$. By the definition of P^0 and P in V and N and the elementariness of k, $M_{\mathcal{U}_x} \models$ "$j_{\mathcal{U}_x}(P) = P * \tilde{R}$ and $\models_P$ 'R is $k(\gamma)^+$ directed closed' ", so $M_{\mathcal{U}_x} \models$ "For some inaccessible $\alpha < j_{\mathcal{U}_x}(\lambda_1)$, $j_{\mathcal{U}_x}(P) = j_{\mathcal{U}_x}(P)_\alpha * \tilde{R}$ and $\Vdash_{j_{\mathcal{U}_x}(P)_\alpha}$ 'R is $k(\gamma)^+$ directed closed' ". Hence, by reflection, $N \models$ "For some inaccessible $\alpha < \lambda_1$, $P = P_\alpha * \tilde{R}$ and $\Vdash_{P_\alpha}$ 'R is γ^+ directed closed' ", so by the closure properties of N, V satisfies the same statement.

Let α_0 be such an α. It must therefore be the case, since $V[G] \models$ "$|\mathrm{TC}(Q)| = \gamma$", that $Q \in V[G_{\alpha_0}]$, where $G_{\alpha_0} = G \restriction \alpha_0$. It is also true, since the above argument can easily be modified to show that for any $\lambda < \kappa$ there must be an α so that $P = P_\alpha * \tilde{R}$ and $\Vdash_{P_\alpha}$ "R is $([\lambda]^{<\kappa})^+$ directed closed", that any subset of $P_\kappa(\lambda)$ in $V[G][G']$ and any ultrafilter on $P_\kappa(\lambda)$ in $V[G][G']$ (where G' is $V[G]$-generic on Q) must already be present in $V[G_\alpha][G']$ for the appropriate $\alpha \geq \alpha_0$, $\alpha < \lambda_1$. Further, it must be the case that for this particular α, $V[G_\alpha] \models$ "Q is κ directed closed".

By the choice of f, for any partial ordering $S \in V[G^0]$ so that $V[G^0] \models$ "$|\mathrm{TC}(S)| < \lambda_1$" there is in V for $x = \langle \tilde{S}, \alpha \rangle$, $\tilde{S}$ a term for S in the forcing language associated with P^0, α as above, a 2 huge ultrafilter $\mathcal{U}_x$ on $P^{\lambda_1}(\lambda_2)$ with sequence $\langle \lambda_0, \lambda_1, \lambda_2 \rangle$ so that $(j_{\mathcal{U}_x} f)(\kappa) = \langle \tilde{S}, \alpha \rangle$. Let $\lambda < \lambda_1$ be any cardinal. If α is so that $\Vdash_{P^0}$ "$\alpha > \max(|\mathrm{TC}(S)|, 2^{[\lambda]^{<\kappa}})$", then Laver's original argument [7] can be used to show that $M_{\mathcal{U}_x} \models$ "P_0 is an initial segment of $j_{\mathcal{U}_x}(P^0)$, $j_{\mathcal{U}_x}(P^0)_{\kappa+1} = P^0 * \tilde{S}$, for $\kappa + 1 \leq \beta < \alpha$, $j_{\mathcal{U}_x}(P^0)_{\beta+1} = j_{\mathcal{U}_x}(P^0)_\beta * \{\varnothing\}$, and $\Vdash_{j_{\mathcal{U}_x}(P^0)_\alpha}$ 'R is α directed closed' ", where $j_{\mathcal{U}_x}(P^0)_\alpha * \tilde{R} = j_{\mathcal{U}_x}(P^0)$. Further, it is the case that in $M_{\mathcal{U}_x}$, $\Vdash_{j_{\mathcal{U}_x}(P^0)}$ "$j_{\mathcal{U}_x}(S)$ is λ_1 directed closed". We can thus use Laver's and Silver's arguments to show that for any $V[G^0]$-generic set H^0 on S there is a $V[G^0][H^0]$-generic set H^1 on the portion of $j_{\mathcal{U}_x}(P^0) * \widetilde{j_{\mathcal{U}_x}(S)}$ defined between stages $\kappa + 2$ and $\lambda_1 + 1$ so that for $p \in G^0 * H^0$, $j_{\mathcal{U}_x}(p) \in G^0 * H^0 * H^1$, i.e., so that $j_{\mathcal{U}_x}(p) \in G^0 * H^0 \subseteq G^0 * H^0 * H^1$. The embedding $j_{\mathcal{U}_x}$ thus extends in $V[G^0 * H^0]$ to $P_\kappa(\lambda)$, enabling us to define a supercompact measure $\mathcal{U}_\lambda$ on $P_\kappa(\lambda)$ by $C \in \mathcal{U}_\lambda \Leftrightarrow \langle j_{\mathcal{U}_x}(\beta) : \beta < \lambda \rangle \in j_{\mathcal{U}_x}(C)$ which, by choice of α, is in $V[G^0][H^0]$. Hence, for $S = T * \tilde{Q}$, where T is so that $P^0 * \tilde{T} = P_\alpha$, $V[G^0][H^0] = V[G_\alpha][G'] \models$ "κ is λ supercompact with $\mathcal{U}_\lambda$ a normal measure on $P_\kappa(\lambda)$". By the results of the preceding paragraph, $V[G][G'] \models$ "$\mathcal{U}_\lambda$ is a normal ultrafilter on $P_\kappa(\lambda)$". Since λ was an arbitrary cardinal $< \lambda_1$, this establishes Theorem 1.1. □ Theorem 1.1

Henceforth, let us take $\bar{V} = V[G]$ as our ground model, and let $j_0 : \bar{V} \to \bar{V}^{P^\kappa(\lambda_1)}/\mathcal{U}_0 = M$ be the huge embedding corresponding to the ultrafilter $\mathcal{U}_0$ of Theorem 1.1. As $\bar{V}^{P^\kappa(\lambda_1)}/\mathcal{U}_0 \models$ "The $< [\overline{p \cap \lambda_1}]_{\mathcal{U}_0}$ supercompactness of $[\overline{p \cap \kappa}]_{\mathcal{U}_0}$ is indestructible under forcing with $[\overline{p \cap \kappa}]_{\mathcal{U}_0}$ directed closed partial orderings Q so that $|\mathrm{TC}(Q)| < [\overline{p \cap \lambda_1}]_{\mathcal{U}_0}$", Los' theorem and the fact that for any $p \in P^\kappa(\lambda_1)$, $\overline{p \cap \lambda_1} = \kappa$ yield $A_1 = \{\alpha < \kappa : \alpha$ is $< \kappa$ supercompact and the $< \kappa$ supercompactness of α is indestructible under forcing with α directed closed partial orderings Q so that $|\mathrm{TC}(Q)| < \kappa\}$ is unbounded in κ. Thus, by the elementariness of j_0 and the closure properties of M, we can let γ_0 be the least ordinal $> \kappa$ so that $\bar{V}$ and M both satisfy "The $< \lambda_1$ supercompactness of γ_0 is indestructible under forcing with γ_0 directed closed partial orderings Q so that $|\mathrm{TC}(Q)| < \lambda_1$".

Using the embedding j_0, we define a Radin sequence of measures $\mu_{<\kappa^+} = \langle \mu_\alpha : \alpha < \kappa^+ \rangle$ on $R(\gamma_0)$ by $\mu_0(x) = 1$ iff $\langle j_0(\beta) : \beta < \gamma_0 \rangle \in j(x)$, and for $0 < \alpha < \kappa^+$, $\mu_\alpha(x) = 1$ iff $\langle \mu_\beta : \beta < \alpha \rangle \in j_0(x)$. $R_{<\kappa^+}$ is supercompact Radin forcing defined using $\mu_{<\kappa^+}$, i.e., $R_{<\kappa^+}$ consists of all finite sequences of the form $\langle \langle p_0, u_0, C_0 \rangle, \ldots, \langle p_n, u_n, C_n \rangle, \langle \mu_{<\kappa^+}, C \rangle \rangle$ with the following properties.

(1) For $i < j \le n$, $p_i \subsetneq p_j$.

(2) For $i \le n$, $\overline{p_i \cap \kappa}$ is a $< \kappa$ supercompact cardinal whose $< \kappa$ supercompactness is indestructible under forcing with $\overline{p_i \cap \kappa}$ directed closed partial orderings Q so that $|\mathrm{TC}(Q)| < \kappa$.

(3) $\bar{p}_i$ is the least cardinal $> \overline{p_i \cap \kappa}$ which is $< \kappa$ supercompact and whose $< \kappa$ supercompactness is indestructible under forcing with $\bar{p}_i$ directed closed partial orderings Q so that $|\mathrm{TC}(Q)| < \kappa$. We adopt Gitik's notation of [4] and write $\bar{p}_i = (\overline{p_i \cap \kappa})^*$.

(4) For $i \le n$, u_i is a Radin sequence of measures on $R(\bar{p}_i)$ with $(u_i)_0$ a supercompact measure on $P_{\overline{p_i \cap \kappa}}(\bar{p}_i)$.

(5) C_i is a sequence of measure 1 sets for u_i.

(6) C is a sequence of measure 1 sets for $\mu_{<\kappa^+}$.

(7) For each $p \in (C)_0$, where $(C)_0$ is the coordinate of C so that $(C)_0 \in \mu_0$, $\bigcup_{i=1}^n p_i \subsetneq p$.

(8) For each $p \in (C)_0$, $\bar{p} = (\overline{p \cap \kappa})^*$ and $\overline{p \cap \kappa}$ is a $< \kappa$ supercompact cardinal whose $< \kappa$ supercompactness is indestructible under forcing with $\overline{p \cap \kappa}$ directed closed partial orderings Q so that $|\mathrm{TC}(Q)| < \kappa$.

Properties (1) and (7) both follow from the fact that μ_0 is a supercompact measure on $P_\kappa(\gamma_0)$. Properties (4), (5), and (6) are all standard properties of Radin forcing. Properties (2), (3), and (8) all follow since μ_0 is generated by j_0, or, equivalently, by $\mathcal{U}_0 \upharpoonright \gamma_0$, so we can assume that each p_i and each p is an element of $\{p \in P^\kappa(\lambda_1)$: $\overline{p \cap \kappa}$ is a $<\kappa$ supercompact cardinal whose $<\kappa$ supercompactness is indestructible under forcing with $\overline{p \cap \kappa}$ directed closed partial orderings Q so that $|\mathrm{TC}(Q)| < \kappa$ and $\overline{p \cap \gamma_0} = (\overline{p \cap \kappa})^*\} \restriction \gamma_0$.

If

$$\pi_0 = \langle \langle p_0, u_0, C_0 \rangle, \ldots, \langle p_n, u_n, C_n \rangle, \langle \mu_{<\kappa^+}, C \rangle \rangle$$

and

$$\pi_1 = \langle \langle q_0, v_0, D_0 \rangle, \ldots, \langle q_m, v_m, D_m \rangle, \langle \mu_{<\kappa^+}, D \rangle \rangle$$

are two conditions in $R_{<\kappa^+}$, then π_1 ext π_0 if the following conditions hold.

(1) For each $\langle p_j, u_j, C_j \rangle$ which appears in π_0 there is a $\langle q_i, v_i, D_i \rangle$ which appears in π_1 so that $\langle q_i, v_i \rangle = \langle p_j, u_j \rangle$ and $D_i \subseteq C_j$.

(2) $D \subseteq C$.

(3) $n \leq m$.

(4) If $\langle q_i, v_i, D_i \rangle$ does not appear in π_0, let $\langle p_j, u_j, C_j \rangle$ (or $\langle \mu_{<\kappa^+}, C \rangle$) be the first element of π_0 so that $\overline{p_j \cap \kappa} > \overline{q_i \cap \kappa}$. Then

(a) q_i is order isomorphic to some $q \in (C_j)_0$.

(b) There exists an $\alpha < \alpha_0$, where α_0 is the length of u_j, so that v_i is isomorphic "in a natural way" to an ultrafilter sequence $v \in (C_j)_\alpha$.

(c) For β_0 the length of v_i, there is a function $f : \beta_0 \to \alpha_0$ so that for $\beta < \beta_0$, $(D_i)_\beta$ is a set of ultrafilter sequences so that for some subset $(D_i)'_\beta$ of $(C_j)_{f(\beta)}$, each ultrafilter sequence in $(D_i)_\beta$ is isomorphic "in a natural way" to an ultrafilter sequence in $(D_i)'_\beta$.

Note that condition (c) is the appropriate modification of Radin's notion [13] of a pair $\langle v, E \rangle$ being in the shadow of $\langle u, E' \rangle$. Note also that the natural isomorphisms discussed above are needed since for each $q \in (C)_0$, any ultrafilter sequence in $(C)_\alpha$ that can be paired with q is actually an ultrafilter sequence appropriate to supercompact Radin forcing on $P_{\overline{q \cap \kappa}}(q)$ which must therefore be identified in the obvious inductive way using the order isomorphism between q and $\bar{q}$ with an ultrafilter sequence appropriate to supercompact Radin forcing on $P_{\overline{q \cap \kappa}}(\bar{q})$. For further information on this and other facts concerning Radin forcing, the reader should consult [3], [4], [11], [13], [15], [16].

We now define a partial ordering P' by

$$P' = R_{<\kappa^+} \times \prod_{\{\langle \alpha, \beta \rangle :\, \alpha < \beta \text{ and } \alpha, \beta \in A_1\}} \mathrm{Col}(\alpha, \beta) \times \prod_{\{\langle \alpha, \beta \rangle :\, \alpha < \beta \text{ and } \alpha, \beta \in A_1\}} \mathrm{SC}(\alpha, \beta)$$

(each $\mathrm{SC}(\alpha, \beta)$ is defined using a normal measure $\mathcal{U}^\beta_\alpha$ on $P_\alpha(\beta)$ which satisfies the Menas partition property) ordered componentwise, and let P be the subordering of P' consisting of all conditions of finite support, also ordered componentwise. Let G be $\bar{V}$-generic on P. The model N_A for Theorem 1, where A is as in the statement of Theorem 1, will be a submodel of $\bar{V}[G]$ and will be Gitik's model N_A of [4]. We describe this model in more detail below.

Let G_0 be the projection of G onto $R_{<\kappa^+}$. For any condition

$$\pi = \langle\langle p_0, u_0, C_0\rangle, \dots, \langle p_n, u_n, C_n\rangle, \langle \mu_{<\kappa^+}, C\rangle\rangle \in R_{<\kappa^+}$$

or any condition

$$\pi = \langle p_0, \dots, p_n, C\rangle \in \mathrm{SC}(\alpha, \beta),$$

call $\langle p_0, \dots, p_n\rangle$ the p-part of π. Let $R = \{p\colon \exists \pi \in G_0[p \in p\text{-part}(\pi)]\}$ and let $R_l = \{p : p \in R$ and p is a limit point of $R\}$. We define three sets E_0, E_1, and E_2 by

$$E_0 = \{\alpha : \text{For some } \pi \in G_0 \text{ and some } p \in p\text{-part}(\pi), \overline{p \cap \kappa} = \alpha\},$$

$$E_1 = \{\alpha : \alpha \text{ is a limit point of } E_0\},$$

and $E_2 = E_1 \cup E_3 = \{\beta\colon \exists \alpha \in E_1[\beta = \alpha^*]\}$. Let $\langle \alpha_\nu\colon \nu < \kappa\rangle$ be the continuous increasing enumeration of E_2, and let $\nu = \nu' + n$ for some $n \in \omega$. For each $\beta \in [\alpha_\nu, \alpha_{\nu+1})$ sets $C_i(\alpha_\nu, \beta)$ are defined according to specific conditions on ν' and ν in the following manner:

(1) $\nu' = \nu \neq 0$ and $n = 0$. Let then $p(\alpha_\nu)$ be the element p of R so that $\overline{p \cap \kappa = \alpha_\nu}$, and let $h_{p(\alpha_\nu)}\colon p(\alpha_\nu) \to \overline{p(\alpha_\nu)}$ be the order isomorphism between $p(\alpha_\nu)$ and $\overline{p(\alpha_\nu)}$.

$$C_1(\alpha_\nu, \beta) = \{h''_{p(\alpha_\nu)} p \cap \beta : p \in R_l, p \subseteq p(\alpha_\nu), \text{and } h^{-1}_{p(\alpha_\nu)}(\beta) \in p\}.$$

(2) $\nu' \neq \nu$ and $n = 2k$. Let $C_2(\alpha_\nu, \beta) = \{h''_{p(\alpha_\nu)} p \cap \beta\colon p \in R$, and if $(\nu' \neq 0)$ or $(\nu' = 0$ and $k > 1)$, then $p(\alpha_{\nu' + 2(k-1)}) \subsetneq p \subseteq p(\alpha_\nu)\}$.

(3) $\nu' \neq \nu$ and $n = 2k + 1$. Let $G(\alpha_\nu, \alpha_{\nu+1})$ be the projection of G onto $\mathrm{SC}(\alpha_\nu, \alpha_{\nu+1})$. $C_3(\alpha_\nu, \beta) = \{p \cap \beta\colon \exists \pi \in G(\alpha_\nu, \alpha_{\nu+1})[p \in p\text{-part}(\pi)]\}$.

(4) $n \neq 0$ or $\nu' = n = 0$. Let $H(\alpha_\nu, \alpha_{\nu+1})$ be the projection of G onto $\mathrm{Col}(\alpha_\nu, \alpha_{\nu+1})$. $C_4(\alpha_\nu, \beta) = H(\alpha_\nu, \alpha_{\nu+1}) \restriction \beta$.

Intuitively, N_A is $R(\kappa)$ of the least model of ZF extending $\bar{V}$ which contains, for each interval $[\alpha_\nu, \alpha_{\nu+1})$ and each $\beta \in [\alpha_\nu, \alpha_{\nu+1})$, $C_1(\alpha_\nu, \beta)$ if ν is a limit ordinal, $C_2(\alpha_\nu, \beta)$ if $\nu = \nu' + 2k$ and $\nu \in B$, $C_3(\alpha_\nu, \beta)$ if $\nu = \nu' + 2k + 1$ and $\nu \in B$, and $C_4(\alpha_\nu, \beta)$ if $\nu \in A \cup \{0\}$. The fact that in $\bar{V}$, each α_ν is a $< \kappa$ supercompact cardinal whose $< \kappa$ supercompactness is indestructible under α_ν directed closed forcing with partial orderings Q so that $|\mathrm{TC}(Q)| < \kappa$, coupled with the homogeneous manner in which N_A is defined will provide the crucial element in showing that α_ν remains a Ramsey cardinal in N_A for $\nu \in A$.

To define N_A more precisely, it is necessary to define canonical names $\underset{\sim}{\alpha}_\nu$ for the α_ν's and canonical names $\underset{\sim}{C_i(\nu,\beta)}$ for the sets $C_i(\alpha_\nu,\beta)$. Recall that it is possible to decide $p(\alpha_\nu)$ (and hence $\overline{p(\alpha_\nu)}$) by writing

$$\omega \cdot \nu = \omega^{\sigma_0} \cdot n_0 + \omega^{\sigma_1} \cdot n_1 + \cdots + \omega^{\sigma_m} \cdot n_m$$

(where $\sigma_0 > \sigma_1 > \cdots > \sigma_m > 0$ are ordinals, $n_0, \dots, n_m > 0$ are integers, and $+, \cdot$, and exponentiation are as in ordinal arithmetic), letting $\pi = \langle\langle p_{ij_i}, u_{ij_i}, D_{ij_i}\rangle_{i \le m, 1 \le j_i \le n_i}, \langle \mu_{<\kappa^+}, D\rangle\rangle$ be so that

$$\min(\overline{p_{i1} \cap \kappa}, \omega^{\mathrm{length}(u_{i1})}) = \sigma_i \quad \text{and} \quad \mathrm{length}(u_{ij_i}) = \min(\overline{p_{i1} \cap \kappa}, \mathrm{length}(u_{i1}))$$

for $1 \le j_i \le n_i$, and letting $p(\alpha_\nu)$ be p_{mn_m}. Further

$$D_\nu = \{r \in P : r \restriction R_{<\kappa^+} \text{ extends a condition } \pi \text{ of the above form}\}$$

is a dense open subset of P. $\underset{\sim}{\alpha}_\nu$ is the name of the α_ν determined by any element of $D_\nu \cap G$; in Gitik's notation of [4], $\underset{\sim}{\alpha}_\nu = \{\langle r, \check{\alpha}_\nu(r)\rangle : r \in D_\nu\}$, where $\alpha_\nu(r)$ is the α_ν determined by the condition r.

The canonical names $\underset{\sim}{C_i(\nu,\beta)}$ for the sets $C_i(\alpha_\nu,\beta)$ are defined in a manner so as to be invariant under the appropriate group of automorphisms. Specifically, there are four cases to consider. We again write $\nu = \nu' + n$ and let $\beta \in [\alpha_\nu, \alpha_{\nu+1})$. We also assume without loss of generality that as in [4], $\alpha_{\nu+1}$ is determined by D_ν.

(1) $\nu' = \nu \neq 0$ and $n = 0$. $\underset{\sim}{C_1(\nu,\beta)}$ is then the name for $\{h''_{p(\alpha_\nu)(r)} p \cap \beta : \exists r \in P[r \in D_\nu \cap G,\ p \in p\text{-part }(r \restriction R_{<\kappa^+}),\ p \subseteq p(\alpha_\nu)(r),\ p \in R_l \restriction r, \text{ and } h^{-1}_{p(\alpha_\nu)(r)}(\beta) \in p\}$ where $p(\alpha_\nu)(r)$ and $h_{p(\alpha_\nu)(r)}$ are the $p(\alpha_\nu)$ and $h_{p(\alpha_\nu)}$ determined by the condition r and $R_l \restriction r$ is the portion of R_l determined by r. Note that this definition is unambiguous, since for any r and r' so that $r, r' \in D_\nu \cap G$, $p(\alpha_\nu)(r) = p(\alpha_\nu)(r')$. In Gitik's notation,

$$\underset{\sim}{C_1(\nu,\beta)} = \{\langle r, (\check{r} \restriction R_{<\kappa^+}) \restriction (\alpha_\nu(r), \beta)\rangle : r \in D_\nu\},$$

where for $r \in P$, $\pi = r \restriction R_{<\kappa^+}$, $\pi \restriction (\alpha_\nu(r), \beta) = \{h''_{p(\alpha_\nu)(r)} p \cap \beta : p \in p\text{-part}(\pi),\ p \subseteq p(\alpha_\nu)(r),\ p \in R_l \restriction \pi, \text{ and } h^{-1}_{p(\alpha_\nu)(r)}(\beta) \in p\}$.

(2) $\nu \notin A$, $\nu' \neq \nu$ and $n = 2k$. Note that as in [4] we can assume without loss of generality that for any $r \in D_\nu$, r determines $\alpha_{\nu'+2(k-1)}$. $\underset{\sim}{C_2(\nu,\beta)}$ is then the name for $\{h''_{p(\alpha_\nu)(r)} p \cap \beta : \exists r \in P[r \in D_\nu \cap G,\ p \in p\text{-part }(r \restriction R_{<\kappa^+}),\ p(\alpha_{\nu'+2(k-1)})(r) \subseteq p \subseteq p(\alpha_\nu)(r),\ p \in R \restriction r, \text{ and } h^{-1}_{p(\alpha_\nu)(r)}(\beta) \in p\}$, where $R \restriction r$ is the portion of R determined by r. The unambiguity of this definition again follows from that fact that for $r, r' \in D_\nu \cap G$,

$$p(\alpha_{\nu'+2(k-1)})(r)=p(\alpha_{\nu'+2(k-1)})(r') \quad \text{and} \quad p(\alpha_\nu)(r)=p(\alpha_\nu)(r').$$

In Gitik's notation,

$$\underline{C_2(\nu,\beta)}=\{\langle r,(\check{r}\restriction R_{<\kappa^+})\restriction(\alpha_\nu(r),\beta)\colon r\in D_\nu\},$$

where this time for $r\in P$, $\pi=r\restriction R_{<\kappa^+}$, $\pi\restriction(\alpha_\nu(r),\beta)=\{h''_{p(\alpha_\nu)(r)}p\cap\beta\colon p\in p\text{-part}(\pi),\ p\in R\restriction\pi,\ p(\alpha_{\nu'+2(k-1)})(r)\subseteq p\subseteq p(\alpha_\nu)(r),\text{ and } h^{-1}_{p(\alpha_\nu)(r)}(\beta)\in p\}$.

(3) $\nu\notin A$, $\nu'\neq\nu$ and $n=2k+1$. $\underline{C_3(\nu,\beta)}$ is then the name for $\{p\cap\beta\colon \exists r\in P[r\in D_\nu\cap G,\ p\in p\text{-part}(r\restriction \mathrm{SC}(\alpha_\nu(r),\alpha_{\nu+1}(r)))]\}$. As before, this definition is unambiguous. In Gitik's notation,

$$\underline{C_3(\nu,\beta)}=\{\langle r,(\check{r}\restriction \mathrm{SC}(\alpha_\nu(r),\alpha_{\nu+1}(r))\restriction(\alpha_\nu(r),\beta)\rangle\colon r\in D_\nu\},$$

where for $r\in P$, $\pi=r\restriction \mathrm{SC}(\alpha_\nu(r),\alpha_{\nu+1}(r))$, $\pi\restriction(\alpha_\nu(r),\beta)=\{p\cap\beta\colon p\in p\text{-part}(\pi)\}$.

(4) $\nu\in A\cup\{0\}$. $\underline{C_4(\nu,\beta)}$ is then the name for $\{p\restriction\beta\colon \exists r\in P[r\in D_\nu\cap G,\ p\in r\restriction \mathrm{Col}(\alpha_\nu(r),\alpha_{\nu+1}(r))]\}$. As before, this definition is unambiguous. In Gitik's notation,

$$\underline{C_4(\nu,\beta)}=\{\langle r,(r\restriction \mathrm{Col}(\alpha_\nu(r),\alpha_{\nu+1}(r)))\restriction\beta\rangle\colon r\in D_\nu\}.$$

Let $\mathscr{G}$ be the group of automorphisms of [4], and let

$$\underline{C(G)}=\bigcup_{i=1}^{4}\{\pi(\underline{C_i(\nu,\beta)})\colon \pi\in\mathscr{G},\ 0<\nu<\kappa,\text{ and }\beta\in[\nu,\kappa)\text{ is a cardinal}\}.$$

$C(G)=\bigcup_{i=1}^4\{i_G(\pi(\underline{C_i(\nu,\beta)}))\colon \pi\in\mathscr{G},\ 0<\nu<\kappa,\text{ and }\beta\in[\nu,\kappa)\text{ is a cardinal}\}=i_G(\underline{C(G)})$. N_A is then the set of all sets of rank $<\kappa$ of the model consisting of all sets which are hereditarily $\bar{V}$ definable from $C(G)$, i.e., $N_A=R(\kappa)^{H\bar{V}D(C(G))}$

Gitik [4] has shown the following facts about N_A.

(1) $N_A\models\forall\nu[\aleph_\nu=\alpha_\nu]$.

(2) $N_A\models \mathrm{ZF}+\neg \mathrm{AC}_\omega$.

In addition to these facts, we know that for any set $x\subseteq\alpha_\nu$ in N_A, α_ν arbitrary,

$$x=\{\alpha<\alpha_\nu\colon \bar{V}[G]\models\phi(\alpha,i_G(\pi_1(\underline{C_{i_1}(\nu_1,\beta_1)})),\dots,i_G(\pi_n(\underline{C_{i_n}(\nu_n,\beta_n)})),C(G))\},$$

where i_j is an integer, $1\le j\le n$, $1\le i_j\le 4$, and $\phi(x_0,\dots,x_{n+1})$ is a formula which may also contain some parameters from $\bar{V}$ which we shall suppress.

Let

$$\bar{P}=\prod_{\{i_j:i_j=4,j\le n\}}\mathrm{Col}(\alpha_{\nu_j},\beta_j)\times\prod_{\{i_j:i_j=3,j\le n\}}\mathrm{SC}(\alpha_{\nu_j},\beta_j)\times R_{<\kappa^+}.$$

For $\pi\in R_{<\kappa^+}$, let $\pi\restriction\alpha_\nu=\{\langle r,u,D\rangle\in\pi\colon \overline{r\cap\kappa}\le\alpha_\nu\}$, and for $p\in\bar{P}$, $p=\langle p_1,\dots,p_m,\pi\rangle$, $m\le n$, $\pi\in R_{<\kappa^+}$, let $p\restriction\alpha_\nu=\langle q_1,\dots,q_m,\pi\restriction\alpha_\nu\rangle$, where $q_j=p_j$ if

$\alpha_{\nu_j} \leq \alpha_\nu$ and $q_j = \varnothing$ otherwise. In other words, $p \restriction \alpha_\nu$ is the part of p below α_ν. Without loss of generality, we ignore the empty coordinates and let $\bar{P} \restriction \alpha_\nu = \{p \restriction \alpha_\nu : p \in \bar{P}\}$. Let $G \restriction \alpha_\nu$ be the projection of G onto $\bar{P} \restriction \alpha_\nu$. Gitik has also shown (Theorem 3.2.11, [4]) the following.

(3) For any $x \subseteq \alpha_\nu$ as in the above, $x \in \bar{V}[G \restriction \alpha_\nu]$. Further, the elements of $\bar{P} \restriction \alpha_\nu$ can be partitioned into $< \alpha_{\nu+1}$ equivalence classes with respect to the sequence $\underline{C_{i_1}(\nu_1, \beta_1), \ldots, C_{i_n}(\nu_n, \beta_n)}$ (the "almost similar" equivalence classes of [4]) so that if $\alpha < \alpha_\nu$, τ is a term for x, and $p \Vdash \underset{\sim}{\alpha} \in \tau$, for any q in the same equivalence class as p, $q \Vdash \underset{\sim}{\alpha} \in \tau$.

Fact (3) above essentially says that any subset of α_ν in N_A is determined by a partial ordering of cardinality $< \alpha_{\nu+1}$. This will be the key fact in the proof of the next three lemmas.

Lemma 1.3. *If $\nu + 1 \in A$, then $N_A \models$ "$\aleph_{\nu+1}$ is a Ramsey cardinal".*

Proof of Lemma 1.3. If $f \in N_A$ is so that $f : [\aleph_{\nu+1}]^{<\omega} \to 2$, then since f can be coded by a subset of $\aleph_{\nu+1}$, facts (1) and (3) above tell us that for some term $\tau(x, y)$ which may also contain elements of $\bar{V}$, $\tau(x, y)$ denotes f in $\bar{V}[G \restriction \alpha_{\nu+1}] = \bar{V}[C_4(\nu + 1, \beta), G \restriction \alpha_\nu]$ for the appropriate $\bar{P} \restriction \alpha_{\nu+1} = \mathrm{Col}(\alpha_{\nu+1}, \beta) \times \bar{P} \restriction \alpha_\nu$. (We will have $\bar{V}[G \restriction \alpha_{\nu+1}] \models$ "$f(t) = i$" iff $\exists p \in G \restriction \alpha_{\nu+1} [p \Vdash \tau(\underset{\sim}{i}, \underset{\sim}{t})]$ where $\underset{\sim}{i}$ is a term for 0 or 1.) Further, as in Theorem 3.2.11, (iii) of [4], if $\langle [p_\gamma] : \gamma < \delta < \alpha_{\nu+1} \rangle$ is an enumeration in $\bar{V}$ of the almost similar equivalence classes of $\bar{P} \restriction \alpha_\nu$ with respect to the sequence $\underline{C_{i_1}(\nu_1, \beta_1), \ldots, C_{i_n}(\nu_n, \beta_n)}$, it is the case that if $\underset{\sim}{t}$ is a term (which, without loss of generality, can be assumed to be in $\bar{V}$) for an arbitrary element t of $[\alpha_{\nu+1}]^{<\omega}$ and $p \in \mathrm{Col}(\alpha_{\nu+1}, \beta)$ is so that for some $q_0 \in \bar{P} \restriction \alpha_\nu$, $\langle q_0, p \rangle \Vdash \tau(\underset{\sim}{0}, \underset{\sim}{t})$, then $\langle q_1, p \rangle \Vdash \tau(\underset{\sim}{0}, \underset{\sim}{t})$ for any q_1 so that q_0 and q_1 are in the same equivalence class. (This is also true if $\langle q_0, p \rangle \Vdash \tau(\underset{\sim}{1}, \underset{\sim}{t})$.) This means that the elements of $\langle [p_\gamma] : \gamma < \delta < \alpha_{\nu+1} \rangle$ completely determine f when forcing over $\bar{V}[C_4(\nu + 1, \beta)]$ with $\bar{P} \restriction \alpha_\nu$. More generally, if x represents any subset of $\alpha_{\nu+1}$ in $\bar{V}[G \restriction \alpha_{\nu+1}]$, then $\langle [p_\gamma] : \gamma < \delta < \alpha_{\nu+1} \rangle$ completely determines x when forcing over $\bar{V}[C_4(\nu + 1, \beta)]$ with $\bar{P} \restriction \alpha_\nu$.

By the definition of the partial ordering P, since $\mathrm{Col}(\alpha_{\nu+1}, \beta)$ is $\alpha_{\nu+1}$ directed closed and $\beta < \kappa$, $\bar{V}[C_4(\nu + 1, \beta)] \models$ "$\alpha_{\nu+1}$ is supercompact". Hence, $\bar{V}[C_4(\nu + 1, \beta)] \models$ "$\alpha_{\nu+1}$ is a measurable cardinal"; therefore, let μ be a fixed normal measure on $\alpha_{\nu+1}$ in $\bar{V}[C_4(\nu + 1, \beta)]$. Since any subset x of $\alpha_{\nu+1}$ is determined by a set of equivalence classes of conditions of cardinality $< \alpha_{\nu+1}$, the Lévy–Solovay arguments [8] show that $\mu' = \{x \subseteq \alpha_{\nu+1} : x$ contains a μ measure 1 set$\}$ is a normal measure on $\alpha_{\nu+1}$ in $\bar{V}[C_4(\nu + 1, \beta), G \restriction \alpha_\nu]$. Thus, Rowbottom's theorem [14] shows that there is a set $x \in \mu'$ which is homogeneous for f; without loss of generality, we can assume that $x \in \bar{V}[C_4(\nu+1,\beta)]$. As the definition of N_A insures that $\bar{V}[C_4(\nu+1,\beta)] \subseteq N_A$, $x \in N_A$. This proves LEMMA 1.3. □ Lemma 1.3

LEMMA 1.4. *If $\nu+1 \in B$, then $N_A \models$ "$\aleph_{\nu+1}$ is a singular Rowbottom cardinal which carries a Rowbottom filter".*

PROOF OF LEMMA 1.4. Let $\nu+1 = \nu' + n_0$ where ν' is a limit ordinal and $0 < n_0 < \omega$. We consider two cases, namely $n_0 = 2k$ and $n_0 = 2k+1$. First, fix f: $[\aleph_{\nu+1}]^{<\omega} \to \gamma$ a Rowbottom partition on $[\aleph_{\nu+1}]^{<\omega}$ in N_A. As before, since f can be coded by a subset of $\alpha_{\nu+1}$, $f \in \bar{V}[G \restriction \alpha_{\nu+1}]$ for the appropriate $\bar{P} \restriction \alpha_{\nu+1}$.

If $n_0 = 2k+1$, then $\bar{P} \restriction \alpha_{\nu+1} = \bar{P} \restriction \alpha_\nu \times \mathrm{SC}(\alpha_{\nu+1},\beta)$ for the appropriate β. Since $\mathrm{SC}(\alpha_{\nu+1},\beta) = \mathrm{SC}(\alpha_{\nu+1},\alpha_{\nu+2}) \restriction \beta$, it will be the case that for each $p \in C_3(\nu+1,\beta)$, $\bar{V} \models$ "$\overline{p \cap \kappa}$ is a measurable cardinal", i.e., for each member p of the supercompact Prikry sequence, $\bar{V} \models$ "$\overline{p \cap \kappa}$ is a measurable cardinal". Thus, since forcing with $\mathrm{SC}(\alpha_{\nu+1},\beta)$ adds no new bounded subsets of $\alpha_{\nu+1}$, $\bar{V}[C_3(\nu+1,\beta)] \models$ "$\overline{p \cap \kappa}$ is a measurable cardinal" if $p \in C_3(\nu+1,\beta)$. Let therefore $\langle \gamma_n : n < \omega \rangle$ be the increasing enumeration of $\{\overline{p \cap \kappa} : p \in C_3(\nu+1,\beta)\}$, and let $\langle \mathcal{U}_n : n < \omega \rangle$ be a sequence so that $\bar{V}[C_3(\nu+1,\beta)] \models$ "$\mathcal{U}_n$ is a normal measure on γ_n". Since $\bar{V}$, $\bar{V}[C_3(\nu+1,\beta)]$, and $\bar{V}[\langle \gamma_n : n < \omega \rangle]$ all have the same bounded subsets of $\alpha_{\nu+1}$, it is the case that $\bar{V}[\langle \gamma_n : n < \omega \rangle] \models$ "$\mathcal{U}_n$ is a normal measure on γ_n"; further, the sequence $\langle \mathcal{U}_n : n < \omega \rangle$ can be chosen so that $\langle \mathcal{U}_n : n < \omega \rangle \in \bar{V}[\langle \gamma_n : n < \omega \rangle]$. Hence, since $\bar{V}[\langle \gamma_n : n < \omega \rangle] \subseteq N_A$, it is possible to define in N_A

$$\mathcal{U}^{N_A}_{\nu+1} = \{x \subseteq \alpha_{\nu+1} : \exists n \forall m \geq n [x \cap \gamma_m \in \mathcal{U}_m]\}.$$

Clearly, $\mathcal{U}^{N_A}_{\nu+1}$ is a filter on $\alpha_{\nu+1}$, and $N_A \models$ "$\alpha_{\nu+1}$ is singular". We show that $N_A \models$ "$\mathcal{U}^{N_A}_{\nu+1}$ is Rowbottom".

First, note that since $\bar{V}[C_3(\nu+1,\beta)] \models$ "$\mathrm{cof}(\alpha_{\nu+1}) = \omega$", a theorem of Prikry [12] shows that $\mathcal{U}^{\bar{V}[C_3(\nu+1,\beta)]}_{\nu+1} = \{x \subseteq \alpha_{\nu+1} : x \in \bar{V}[C_3(\nu+1,\beta)]$ and $\exists n \forall m \geq n [x \cap \gamma_m \in \mathcal{U}_m]\}$ is in $\bar{V}[C_3(\nu+1,\beta)]$ a Rowbottom filter on $\alpha_{\nu+1}$. Also, since the generic projection on $\mathrm{SC}(\alpha_{\nu+1},\beta)$ of G, $G(\alpha_{\nu+1},\alpha_{\nu+2}) \restriction \beta$ is so that $\bar{V}[C_3(\nu+1,\beta)] = \bar{V}[G(\alpha_{\nu+1},\alpha_{\nu+2}) \restriction \beta]$ (the generic set is canonically definable from the generic sequence), as in the previous lemma we know that f is completely determined by $\langle [p_\sigma] : \sigma < \delta < \alpha_{\nu+1} \rangle$ when forcing over $\bar{V}[C_3(\nu+1,\beta)]$ with $\bar{P} \restriction \alpha_\nu$. Thus, the Lévy–Solovay results again imply that $\bar{V}[C_3(\nu+1,\beta), G \restriction \alpha_\nu] = \bar{V}[G \restriction \alpha_{\nu+1}] \models$ "$\alpha_{\nu+1}$ is a Rowbottom cardinal and any Rowbottom partition has a homogeneous set $x \in \mathcal{U}^{\bar{V}[C_3(\nu+1,\beta)]}_{\nu+1}$". (An n so that $\forall m \geq n [x \cap \gamma_m \in \mathcal{U}_m]$ will be so that $\gamma_n > \delta$, δ as immediately above.) Since

$\bar{V}[C_3(\nu+1,\beta)]\subseteq N_A$, $\mathcal{U}_{\nu+1}^{\bar{V}[C_3(\nu+1,\beta)]}\subseteq \mathcal{U}_{\nu+1}^{N_A}$, so any Rowbottom partition f: $[\alpha_{\nu+1}]^{<\omega}\to\gamma$ has a homogeneous set $x\in\mathcal{U}_{\nu+1}^{N_A}$.

If $n_0=2k$, then $\bar{P}\upharpoonright\alpha_{\nu+1}=\bar{P}\upharpoonright\alpha_\nu\times R(\alpha_{\nu'+2(k-1)},\alpha_{\nu'+2k})$, where

$$R(\alpha_{\nu'+2(k-1)},\alpha_{\nu'+2k})=\hat{R}=\{\langle r,u,D\rangle\in R_{<\kappa^+}\colon \alpha_{\nu'+2(k-1)}<\overline{r\cap\kappa}\le\alpha_{\nu'+2k}\},$$

i.e., $R(\alpha_{\nu'+2(k-1)},\alpha_{\nu'+2k})$ is the portion of the Radin forcing between $\alpha_{\nu'+2(k-1)}$ and $\alpha_{\nu'+2k}$. Let h be the $h_{q(\alpha_{\nu'+2k})}$ for the q which determines $\alpha_{\nu'+2k}$. For any $\langle q,u,D\rangle$ so that $\overline{q\cap\kappa}=\alpha_{\nu'+2k}$, as in [4] it must be the case that length$(u)=1$. Hence, $\hat{R}$ must be isomorphic to a supercompact Prikry partial ordering on $P_{\alpha_{\nu'+2k}}(\alpha_{\nu'+2k+1})$; in particular, forcing with $\hat{R}$ will add no new bounded subsets to $\alpha_{\nu'+2k}$, the generic sequence $\langle h''p\cap\beta\colon h''p\cap\beta\in C_2(\nu'+2k,\beta)\rangle$ will code a cofinal ω sequence, and for any $\beta\in[\alpha_{\nu'+2k},\alpha_{\nu'+2k+1})$ and any p so that $h''p\cap\beta\in C_2(\nu'+2k,\beta)$, $\bar{V}[\hat{G}]\models$"$(h''p\cap\beta)\cap\kappa=\overline{p\cap\kappa}$ is a measurable cardinal", where $\hat{G}$ is the projection of the generic set G onto $\hat{R}$. (The facts that h is the order isomorphism of q onto $\bar{q}$ and $q\cap\kappa$ is an ordinal $<\kappa$ imply that $(h''p\cap\beta)\cap\kappa=p\cap\kappa=\overline{p\cap\kappa}$.) Thus, let $\langle\gamma_n\colon n<\omega\rangle$ be the sequence which enumerates in increasing order $\langle(h''p\cap\beta)\cap\kappa\colon h''p\cap\beta\in C_2(\nu'+2k,\beta)\rangle$, and let $\langle\mathcal{U}_n\colon n<\omega\rangle$ and $\langle\mathcal{W}_n\colon n<\omega\rangle$ be a sequence of normal measures and of well orderings definable in $\bar{V}[\langle\gamma_n\colon n<\omega\rangle]\subseteq\bar{V}[\hat{G}]$ so that in $\bar{V}$, $\bar{V}[\langle\gamma_n\colon n<\omega\rangle]$, or $\bar{V}[\hat{G}]$, $\mathcal{U}_n$ is a normal measure on γ_n and $\mathcal{W}_n$ well orders $\mathcal{U}_n$. As $\bar{V}[\langle\gamma_n\colon n<\omega\rangle]\subseteq N_A$, $N_A\models$"$\alpha_{\nu'+2k}$ is singular", and we can again define the filter $\mathcal{U}_{\nu'+2k}^{N_A}$ in N_A by

$$\mathcal{U}_{\nu'+2k}^{N_A}=\{x\subseteq\alpha_{\nu'+2k}\colon\exists n\forall m\ge n[x\cap\gamma_m\in\mathcal{U}_m]\}.$$

Since f will be determined by forcing over $\bar{V}[\hat{G}]$ with $\bar{P}\upharpoonright\alpha_\nu=\langle[p_\sigma]\colon\sigma<\delta<\alpha_{\nu+1}\rangle$ (we use the fact that $\alpha_{\nu+1}=\alpha_{\nu'+2k}$), the Lévy–Solovay arguments again imply that $\bar{V}[\hat{G},G\upharpoonright\alpha_\nu]=\bar{V}[G\upharpoonright\alpha_{\nu+1}]\models$"$\langle\mathcal{U}'_n\colon m_0\le n<\omega\rangle$ is a sequence so that $\mathcal{U}'_n=\{x\subseteq\gamma_n\colon x$ contains a $\mathcal{U}_n$ measure 1 set$\}$ is a normal measure on γ_n", where m_0 is the least integer so that $\gamma_{m_0}>\delta$, δ as immediately above.

Prikry's construction [12] of a homogeneous set x for f in $\bar{V}[G\upharpoonright\alpha_{\nu+1}]$ so that $\exists m\ge m_0\forall n\ge m[x\cap\gamma_n\in\mathcal{U}'_n]$ involves inductively defining a sequence $\langle x_n\colon m_0\le n<\omega\rangle$ of sets so that $x_n\in\mathcal{U}'_n$ and so that $\bigcup_{n\in\omega}x_n=x$. The construction of x_{n+1} is accomplished by choosing a set based on x_n, the partition f, the partition $f\upharpoonright[\gamma_{n+1}-\gamma_n]^{<\omega}$, and certain partitions canonically defined using f and $f\upharpoonright[\gamma_{n+1}-\gamma_n]^{<\omega}$. Since we can assume that $x_n\in\mathcal{U}_n$, the choice can be made by using the well ordering $\mathcal{W}_{n+1}$ to pick the appropriate homogeneous sets and hence is absolute given the partition f and the sequences $\langle\gamma_n\colon m_0\le n<\omega\rangle$, $\langle\mathcal{U}_n\colon m_0\le n<\omega\rangle$, and $\langle\mathcal{W}_n\colon m_0\le n<\omega\rangle$. Since each of these three sequences is in N_A, the set x can be constructed working in N_A. Thus, $x \in N_A$, $x \in \mathcal{U}^{N_A}_{\nu'+2k}$, and x is homogeneous for f. This proves Lemma 1.4. □ Lemma 1.4

LEMMA 1.5. *If ν is so that $N_A \models$ "ν is a limit ordinal", then $N_A \models$ "$\aleph_\nu$ is a Jonsson cardinal which carries a Jonsson filter".*

PROOF OF LEMMA 1.5. Let $f \in N_A$ be so that $N_A \models$ "f: $[\aleph_\nu]^{<\omega} \to \aleph_\nu$ is a Jonsson partition". Since $\nu \leq \aleph_\nu = \alpha_\nu$ and f can be coded by a subset of α_ν, it will be the case for the appropriate $\bar{P}\upharpoonright\alpha_\nu$ that $f \in \bar{V}[G\upharpoonright\alpha_\nu]$ and $\bar{V}[G\upharpoonright\alpha_\nu] \models$ "α_ν is a singular cardinal". Further, $\bar{P}\upharpoonright\alpha_\nu$ can be factored into $R_{<\kappa^+}\upharpoonright\alpha_\nu \times Q$, where Q is a product of partial orderings of the form $\mathrm{Col}(\alpha_{\nu_i}, \beta_i)$ and $\mathrm{SC}(\alpha_{\nu_i}, \beta_i)$ so that each α_{ν_i} and each β_i is less than α_ν and $R_{<\kappa^+}\upharpoonright\alpha_\nu = R' = \{q\upharpoonright\alpha_\nu : q \in R_{<\kappa^+}\}$. It is thus the case that $|Q| < \alpha_\nu$.

Let G' be the projection of G onto R'. As R' is the portion of $R_{<\kappa^+}$ through α_ν, G' will contain a Radin generic sequence through α_ν, i.e., $E_4 = \{\alpha < \alpha_\nu : \alpha \in E_0\}$ and $E_5 = \{\alpha < \alpha_\nu : \alpha \in E_1\}$ will both be Radin generic sequences through α_ν which witness the singularity of α_ν. By the definition of N_A, $E_5 \in N_A$.

Let $\langle \sigma_\eta : \eta < \nu \rangle$ be the continuous increasing enumeration of $\{\alpha \in E_5 : \alpha > |Q|\}$. It is then possible to define the sequences $\langle \beta_n : \eta < \nu \rangle$, $\langle \gamma_\eta : \eta < \nu \rangle$, $\langle \mathcal{U}_\eta : \eta < \nu \rangle$, and $\langle \mathcal{W}_\eta : \eta < \nu \rangle$ in $\bar{V}[E_5]$ by

$$\beta_\eta = \begin{cases} \sigma^*_\eta & \text{if } \sigma_\eta = \alpha_{\eta'+2k} \text{ for } 0 \leq k < \omega \text{ and some limit ordinal } \eta', \\ \sigma_\eta & \text{if } \sigma_\eta = \alpha_{\eta'+2k+1} \text{ for } 0 \leq k < \omega \text{ and some limit ordinal } \eta', \end{cases}$$

$\gamma_\eta =$ the least measurable cardinal in $\bar{V} > \beta_\eta$, $\mathcal{U}_\eta =$ a normal measure in $\bar{V}$ on γ_η, and $\mathcal{W}_\eta =$ a well ordering in $\bar{V}$ of $\mathcal{U}_\eta$. It will now be the case that $\bar{V}[G\upharpoonright\alpha_\nu] \models$ "γ_η is a measurable cardinal and $\mathcal{U}'_\eta = \{x \subseteq \gamma_\eta : x$ contains a $\mathcal{U}_\eta$ measure 1 set$\}$ is a normal measure on γ_η". To see this let, for $q \in R'$,

$$q\upharpoonright\eta = \{\langle r, u, D\rangle \in q : \overline{r \cap \kappa} \leq \sigma_\eta\} \quad \text{and} \quad q^\eta = \{\langle r, u, D\rangle \in q : \overline{r \cap \kappa} > \sigma_\eta\}.$$

This allows us to write $R' = R'_\eta \times R^\eta$, where $R'_\eta = \{q\upharpoonright\eta : q \in R'\}$ and $R^\eta = \{q^\eta : q \in R'\}$; further, it allows us to write $G' = G'_\eta \times G^\eta$, where $G'_\eta = \{q\upharpoonright\eta : q \in G'\}$ and $G^\eta = \{q^\eta : q \in G'\}$. By the definition of γ_η and β_η, $|R'_\eta| < 2^{2^{\beta_\eta}} < \gamma_\eta$. Also, for each $q \in R^\eta$ and each $\langle r, u, D\rangle \in q$, the definition of $R_{<\kappa^+}$ insures that $\overline{r \cap \kappa} > \gamma_\eta$; hence, each ultrafilter in the ultrafilter sequence u will be at least γ^+_η additive. Thus, since R^η is a Radin forcing partial ordering, it must have the Prikry property, i.e., for any formula ϕ in the forcing language associated with R^η and any $q \in R^\eta$ it is possible to extend q to a condition q' so that q' decides ϕ only by shrinking the measure 1 sets present in q. The usual Prikry argument then yields that $\bar{V}[G^\eta]$ and $\bar{V}$ have the same subsets of γ_η. Since $|R'_\eta \times Q| < \gamma_\eta$ in both $\bar{V}$ and $\bar{V}[G^\eta]$, the Lévy–Solovay arguments yield that in the model obtained by forcing over $\bar{V}[G^\eta]$ with $R'_\eta \times Q$, i.e., in $\bar{V}[G \restriction \alpha_\nu]$, γ_η is a measurable cardinal and $\mathcal{U}'_\eta$ is a normal measure on γ_η.

Define in N_A $\mathcal{U}_\nu^{N_A} = \{x \subseteq \alpha_\nu : \exists \delta \forall \eta \geq \delta [x \cap \gamma_\eta \in \mathcal{U}_\eta]\}$. Note that $\mathcal{U}_\nu^{N_A} \in N_A$ since $\langle \mathcal{U}_\eta : \eta < \nu \rangle \in \bar{V}[E_5] \subseteq N_A$. $\mathcal{U}_\nu^{N_A}$ is clearly a filter. To see that $N_A \models$ "$\mathcal{U}_\nu^{N_A}$ is a Jonsson filter", first note that Prikry's theorem of [12] also states that it is possible to construct in $\bar{V}[G \restriction \alpha_\nu]$ a homogeneous set x for f so that $\exists \delta \forall \eta \geq \delta [x \cap \gamma_\eta \in \mathcal{U}'_\eta]$. As in Lemma 1.4, it is possible to replace each $\mathcal{U}'_\eta$ with $\mathcal{U}_\eta$. Further, as in Lemma 1.4, the construction of x can be accomplished via an inductive construction of a sequence $\langle x_\eta : \eta < \nu \rangle$ so that $\bigcup_{\eta < \nu} x_\eta = x$ which uses the partition f, the partitions $f \restriction [\gamma_\eta - \bigcup_{\alpha < \eta} \gamma_\alpha]^{<\omega}$, the well ordering $\mathcal{W}_\eta$ to choose the $\mathcal{U}_\eta$ measure 1 set used to construct x_η, and certain partitions which are canonically defined in terms of f and $f \restriction [\gamma_\eta - \bigcup_{\alpha < \eta} \gamma_\alpha]^{<\omega}$. As before, this construction can be carried out in any model of ZF which contains $\langle \gamma_\eta : \eta < \nu \rangle$, $\langle \mathcal{U}_\eta : \eta < \nu \rangle$, and $\langle \mathcal{W}_\eta : \eta < \nu \rangle$. Since $\langle \gamma_\eta : \eta < \nu \rangle$, $\langle \mathcal{U}_\eta : \eta < \nu \rangle$, and $\langle \mathcal{W}_\eta : \eta < \nu \rangle \in \bar{V}[E_5] \subseteq N_A$, x is definable in N_A. As $x \in \mathcal{U}_\nu^{N_A}$ and x is homogeneous for f, Lemma 1.5 is proven. □ Lemma 1.5

Lemma 1.1, Theorem 1.2, and Lemmas 1.3–1.5 complete the proof of Theorem 1. □ Theorem 1

We remark that Prikry's construction of [12] actually shows that $\mathcal{U}_\nu^{N_A}$ is a γ-Rowbottom filter, where $\gamma = \mathrm{cof}(\alpha_\nu)$.

In conclusion, we note that, as pointed out to us by Moti Gitik, it is possible to derive the conclusions of Theorem 1 from an almost huge cardinal κ instead of a 3 huge cardinal. A sketch of the argument is as follows. Since as in [4] the model N_A can be constructed using an almost huge cardinal κ, it suffices to show that if $V \models$ "κ is almost huge" and $j: V \to M$ is an almost huge embedding with $j(\kappa) = \lambda_1$, then it is possible to generically extend V so that $j^*: V[G] \to M[G * H]$ is an almost huge embedding extending j for $H \in V[G]$ and $V[G] \models$ "The $< \lambda_1$ supercompactness of κ is indestructible under forcing with κ directed closed partial orderings Q so that $|\mathrm{TC}(Q)| < \lambda_1$". To do this, note that since $V \models$ "κ is $< \lambda_1$ supercompact and λ_1 is strongly inaccessible", as in [1] or [7] it is possible to show that $V \models$ "There exists a function $f: \kappa \to R(\kappa)$ so that for every x with $|\mathrm{TC}(x)| \leq \lambda < \lambda_1$ there is a supercompact ultrafilter $\mathcal{U}$ on $P_\kappa(\lambda)$ so that $j_{\mathcal{U}}(f)(\kappa) = x$". Using this f, define P^0 as in Theorem 1.1, and let

$P = P^0 * \tilde{P}^1$, where $\tilde{P}^1$ is a term for the portion of $j(P)$ defined in M between κ and λ_1. Let G be V-generic on P. As in [1], [7], or Theorem 1.1, it follows that $V[G] \models$ "κ is $<\lambda_1$ supercompact and the $<\lambda_1$ supercompactness of κ is indestructible under forcing with κ directed closed partial orderings Q so that $|\mathrm{TC}(Q)| < \lambda_1$".

To see that $V[G] \models$ "κ is almost huge", first note that in M, $j(P) = P^0 * \tilde{P}^1 * \tilde{P}^2 = P * \tilde{P}^2$, where $\tilde{P}^2$ is a term for the portion of $j(P)$ defined in M between λ_1 and $j(\lambda_1) = \lambda_2$ using $j(f)$. As $V \models$ "κ is almost huge", the inner model M can be chosen so that $V \models$ "$|\lambda_2| = \lambda_1$". This allows us to carry out in $V[G]$ an inductive construction in λ_2 stages as follows. Let $P_0 = H_0 = \{\phi\}$. For λ a limit ordinal, if $\langle P_\alpha : \alpha < \lambda \rangle$ and $\langle H_\alpha : \alpha < \lambda \rangle$ are the portions of P^2 and H defined through stage λ, then P_λ and H_λ are either the inverse or direct limit of $\langle P_\alpha : \alpha < \lambda \rangle$ and $\langle H_\alpha : \alpha < \lambda \rangle$ (as calculated in M), the type of limit depending upon the nature of λ in M. At successor stages $\alpha + 1$, let $Q_{\alpha+1}$ in $M[G][H_\alpha]$ be so that $P_{\alpha+1} = P_\alpha * \tilde{Q}_{\alpha+1}$. As $V \models$ "$|P| = \lambda_1$", $V[G] \models$ "$|G| = \lambda_1$"; further, since there is an analogous inductive sequence $\langle R_\alpha : \alpha < \lambda_1 \rangle$ which defines P^1 in $V[G^0]$, and since $V[G] \models$ "$|R_\alpha| < \lambda_1$" for each $\alpha < \lambda_1$, for each $\alpha + 1 < \lambda_2$, $V[G] \models$ "$S_{\alpha+1} = \{r : \exists p \in G[j(p) = q * r$ and $r \in Q_{\alpha+1}]\}$ has cardinality $< \lambda_1$". Therefore, since in $M \Vdash_P$ "Each $P_{\alpha+1}$ is λ_1 directed closed" and $M^{<\lambda_1} \subseteq M$, $V[G] \models$ "Each $Q_{\alpha+1}$ is λ_1 directed closed". Thus, in $V[G]$ let $s_{\alpha+1} \in Q_{\alpha+1}$ extend each s in $S_{\alpha+1}$. Since we can assume that $M[G][H_\alpha] \models$ "$|Q_{\alpha+1}| < \lambda_2$", we can let $D = \langle D_\beta : \beta < \lambda_1 \rangle$ be an enumeration in $V[G]$ of the dense open subsets of $Q_{\alpha+1}$ found in $M[G][H_\alpha]$. We can then construct a sequence $\langle q_\beta : \beta < \lambda_1 \rangle$ in $V[G]$ so that $q_0 \in D_0$, q_0 ext $s_{\alpha+1}$, for each $\beta < \lambda_1$, $q_\beta \in D_\beta$, and for each $\gamma < \beta$, q_β ext q_γ. This in turn allows us to define in $V[G]$ the set $H'_{\alpha+1} = \{p : \exists \beta < \lambda_1 [q_\beta \text{ ext } p]\}$, a set which can be verified to be $M[G][H_\alpha]$-generic on $Q_{\alpha+1}$. The set $H_{\alpha+1} = H_\alpha * H'_{\alpha+1}$ is then $M[G]$-generic on $P_{\alpha+1}$. If we let H be the direct limit of $\langle H_\alpha : \alpha < \lambda_2 \rangle$, then it is the case that H is $M[G]$-generic on P^2 and $j''G \subseteq G * H$. We can now define j^* as in Theorem 1.1 and show that j^* extends j and $M[G * H]^{<\lambda_1} \subseteq M[G * H]$, thus showing that κ is almost huge in $V[G]$.

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